1. State whether the given p-series converges.

(a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2}}$ $= \sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$

$\frac{2}{3} < 1$ so $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^2}}$ diverges

(b) $\sum_{n=1}^{\infty} \frac{n^e}{n^\pi}$ $= \sum_{n=1}^{\infty} \frac{1}{n^{\pi - e}}$

$\pi - e \approx 3.1 - 2.7 = 0.4 < 1$

so $\sum_{n=1}^{\infty} \frac{n^e}{n^\pi}$ diverges

2. Use the integral test to determine whether the following sums converge.

(a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+5}}$

Note: $\frac{1}{\sqrt{x+5}}$ is continuous, decreasing, and positive for $x \geq 1$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+5}} \geq \int_1^{\infty} \frac{1}{\sqrt{x+5}}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{\sqrt{x+5}}\,dx$$

$$= \lim_{t\to\infty} 2\sqrt{x+5}\,\Big|_1^t$$

$$= \lim_{t\to\infty} 2\sqrt{t+5} - 2\sqrt{6}$$

$$= \infty$$

$\to \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+5}}$ diverges

(b) $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$

Note: $\frac{1}{x(\ln(x))^2}$ is continuous, decreasing, and positive

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2} \leq \int_2^{\infty} \frac{1}{x(\ln(x))^2}\,dx = \lim_{t\to\infty} \int_2^t \frac{1}{x(\ln x)^2}\,dx$$

$u = \ln(x)$, $du = \frac{1}{x}\,dx$

$$= \lim_{t\to\infty} \int_{\ln(2)}^{\ln(t)} \frac{1}{u^2}\,du$$

$$= \lim_{t\to\infty} -\frac{1}{u}\Big|_{\ln(2)}^{\ln(t)}$$

$$= \lim_{t\to\infty} \frac{-1}{\ln(t)} + \frac{1}{\ln(2)}$$

$$= \frac{1}{\ln(2)}$$

$\to \sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$ converges

(c) $\sum_{n=1}^{\infty} \frac{n}{1+n^2}$

Note: $\frac{x}{1+x^2}$ is positive, continuous, and decreasing (since $\frac{d}{dx}\left(\frac{x}{1+x^2}\right) = \frac{(1+x^2)-x(2x)}{(1+x^2)^2} = \frac{1-x^2}{(1+x^2)^2} < 0$ for $x>1$)

$$\to \sum_{n=1}^{\infty} \frac{n}{1+n^2} \geq \int_2^{\infty} \frac{x}{1+x^2}\,dx = \lim_{t\to\infty} \int_2^t \frac{x}{1+x^2}\,dx = \lim_{t\to\infty} \int_5^{1+t^2} \frac{1}{u}\,\frac{du}{2}$$

$u = 1+x^2$, $\frac{du}{2} = x\,dx$

$$= \lim_{t\to\infty} \ln|u| \Big|_5^{1+t^2} = \lim_{t\to\infty} \ln|1+t^2| - \ln|5| = \infty$$

$\to \sum \frac{n}{1+n^2}$ diverges

(d) $\sum_{n=1}^{\infty} ne^{-n}$

Note xe^{-x} is positive, continuous, and decreasing $\left(\frac{d}{dx}(xe^{-x}) = e^{-x} - xe^{-x} = (1-x)e^{-x} < 0 \text{ for } x>1\right)$

$$\sum_{n=1}^{\infty} ne^{-n} \leq \int_1^{\infty} xe^{-x}\,dx = \lim_{t\to\infty} \int_1^t xe^{-x}\,dx = \lim_{t\to\infty} -xe^{-x}\Big|_1^t + \int_1^t e^{-x}\,dx = \lim_{t\to\infty} -te^{-t} + e^{-1} - e^{-x}\Big|_1^t$$

$u = x \quad dv = e^{-x}dx$

$du = dx \quad v = -e^{-x}$

$$= \lim_{t\to\infty} -te^{-t} + e^{-1} - e^{-t} + e^{-1} = 2e^{-1}$$

$\to \sum_{n=1}^{\infty} ne^{-n}$ converges

3. Use your favorite technological tool (e.g. Wolfram|Alpha) to compute the given partial sum. Then use the estimate $R_N \leq \int_N^{\infty} f(x)\,dx$ to find a bound for the remainder.

(a) $\sum_{n=1}^{1000} \frac{1}{n^3} \approx 1.202$

$$R_{1000} \leq \int_{1000}^{\infty} \frac{1}{x^3}\,dx = \lim_{t\to\infty} \int_{1000}^t \frac{1}{x^3}\,dx = \lim_{t\to\infty} \frac{x^{-2}}{-2}\Big|_{1000}^t = \lim_{t\to\infty} \frac{-1}{2t^2} - \frac{-1}{2\cdot 1000^2}$$

$$= \frac{1}{2\cdot 1000000}$$

(b) $\sum_{n=1}^{100} ne^{-n} \approx 0.9207$

$$R_{100} \leq \int_{100}^{\infty} xe^{-x}\,dx = \lim_{t\to\infty} -te^{-t} + 100e^{-100} - e^{-t} + e^{-100}$$

$$= 101e^{-100}$$

see 2d

4. Find the minimum value of N such that the remainder estimate $R_N < \int_N^\infty f(x)\,dx$ guarantees that $\sum_{n=1}^{N} a_n$ estimates $\sum_{n=1}^{\infty} a_n$ within the given level of accuracy. Then compute the estimate $\sum_{n=1}^{N} a_n$.

(a) $a_n = \frac{1}{n^{1.2}}$, with error less than 0.1

Want $R_{n_0} < 0.1$

Goal: $\int_{n_0}^{\infty} \frac{1}{x^{1.2}}\,dx < 0.1$

LHS: $\int_{n_0}^{\infty} \frac{1}{x^{1.2}}\,dx = \lim_{t\to\infty} \int_{n_0}^{t} \frac{1}{x^{1.2}}\,dx = \lim_{t\to\infty} \frac{x^{-0.2}}{-0.2}\Big|_{n_0}^{t}$

$= \lim_{t\to\infty} -5\,t^{-0.2} + 5n_0^{-0.2} = 5n_0^{-0.2}$

Goal: $5n_0^{-0.2} < 0.1 \iff 50 < n_0^{0.2} \iff 50^5 < n_0$

$n_0 = 50^5 + 1$: $\sum_{n=1}^{50^5+1} \frac{1}{n^{1.2}} \approx 5.492$

(b) $a_n = \frac{1}{1+n^2}$, with error less than 10^{-6}

Want: $R_{n_0} < 10^{-6}$

Goal: $\int_{n_0}^{\infty} \frac{1}{1+x^2}\,dx < 10^{-6}$

LHS: $\int_{n_0}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \arctan(x)\Big|_{n_0}^{t} = \lim_{t\to\infty} \arctan(t) - \arctan(n_0)$

$= \frac{\pi}{2} - \arctan(n_0)$

Goal: $\frac{\pi}{2} - \arctan(n_0) < 10^{-6} \iff \frac{\pi}{2} - 10^{-6} < \arctan(n_0)$

$\iff \tan\left(\frac{\pi}{2} - 10^{-6}\right) < n_0$

$\to 999999.9$

$n_0 = 1000000$ $\sum_{n=1}^{1000000} \frac{1}{1+n^2} \approx 1.07667$

5. Find the values of p for which the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ is convergent.

note: $\frac{1}{x(\ln(x))^p}$ is continuous and positive for $x > 1$

$\frac{1}{x\ln(x)^p}$ is decreasing for $x > 1$ when $p > 0$

$\sum_{n=2}^{\infty} \frac{1}{n\ln(n)^p}$ converges if and only if $\int_2^{\infty} \frac{1}{x\ln(x)^p}\,dx$ converges

$$\int_2^{\infty} \frac{1}{x\ln(x)^p}\,dx = \lim_{t\to\infty} \int_2^t \frac{1}{x\ln(x)^p}\,dx = \lim_{t\to\infty} \int_{\ln(2)}^{\ln(t)} \frac{1}{u^p}\,du$$

$u = \ln(x)$

$du = \frac{1}{x}dx$

$$\lim_{t\to\infty} \int_{\ln(2)}^{\ln(t)} \frac{1}{u^p}\,du = \begin{cases} \lim_{t\to\infty} \ln|u|\Big|_{\ln(2)}^{\ln(t)} & \text{if } p = 1 \\ \lim_{t\to\infty} \frac{u^{1-p}}{1-p}\Big|_{\ln(2)}^{\ln(t)} & \text{if } p \neq 1 \end{cases}$$

$$= \begin{cases} \lim_{t\to\infty} \ln|\ln(t)| - \ln(\ln(2)) & \text{if } p = 1 \\ \lim_{t\to\infty} \frac{\ln(t)^{1-p} - \ln(2)^{1-p}}{1-p} & \text{if } p \neq 1 \end{cases}$$

$$= \begin{cases} \infty & \text{if } p = 1 \\ \frac{-\ln(2)^{1-p}}{1-p} & \text{if } p > 1 \\ \infty & \text{if } p < 1 \end{cases} \rightarrow \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} \text{ converges for } p > 1$$

6. Use the direct comparison test to determine whether each of the following series converge.

(a) $\sum_{n=1}^{\infty} \frac{2}{n(n+1)}$

$0 \le \frac{2}{n(n+1)} \le \frac{2}{n^2} \to 0 \le \sum_{n=1}^{\infty} \frac{2}{n(n+1)} \le \sum_{n=1}^{\infty} \frac{2}{n^2}$

$= 2\sum_{n=1}^{\infty} \frac{1}{n^2}$

$2\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-test, so

$\sum_{n=1}^{\infty} \frac{2}{n(n+1)}$ also converges

(b) $\sum_{n=2}^{\infty} \frac{n^3}{n^4-1}$

$\frac{n^3}{n^4-1} \ge \frac{n^3}{n^4} = \frac{1}{n}$

$\to \sum_{n=2}^{\infty} \frac{n^3}{n^4-1} \ge \sum_{n=2}^{\infty} \frac{1}{n}$

$\sum_{n=2}^{\infty} \frac{1}{n}$ diverges by the p-test, so

$\sum_{n=2}^{\infty} \frac{n^3}{n^4-1}$ also diverges

(c) $\sum_{n=1}^{\infty} \frac{n-1}{n \cdot 4^n}$

$0 \le \frac{n-1}{n} \le 1$, so $0 \le \frac{n-1}{n \cdot 4^n} \le \frac{1}{4^n}$

Since $\frac{1}{4^x}$ is positive, continuous and decreasing,

$\sum_{n=1}^{\infty} \frac{1}{4^n}$ converges if and only if $\int_1^{\infty} \frac{1}{4^x} dx$ converges

But $\int_1^{\infty} \frac{1}{4^x} dx = \lim_{t \to \infty} \frac{-4^{-x}}{\ln(4)} \Big|_1^t = \lim_{t \to \infty} \frac{-4^{-t}}{\ln(4)} + \frac{1}{4\ln(4)}$

$= \frac{1}{4\ln(4)}$

Hence, $\sum_{n=1}^{\infty} \frac{1}{4^n}$ converges, so $\sum_{n=1}^{\infty} \frac{n-1}{n4^n}$ also converges

(d) $\sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^2+1}$

$0 \le \frac{\cos^2(n)}{n^2+1} \le \frac{1}{n^2+1}$

$\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ converges by the integral test

so $\sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^2+1}$ also converges.